

Pairing of bosons in the condensed state of the boson-fermion model

A.S. Alexandrov^a

Department of Physics, Loughborough University, Loughborough LE11 3TU, UK

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Abstract. A two component model of negative U centers coupled with the Fermi sea of itinerant fermions is discussed in connection with high-temperature superconductivity of cuprates, and superfluidity of atomic fermions. We examine the phase transition and the condensed state of this boson-fermion model (BFM) beyond the ordinary mean-field approximation in two and three dimensions. No pairing of fermions and no condensation are found in two-dimensions for any symmetry of the order parameter. The expansion in the strength of the order parameter near the transition yields no linear homogeneous term in the Ginzburg-Landau-Gor'kov equation and a zero upper critical field in *any*-dimensional BFM, which indicates that previous mean-field discussions of the model are flawed. Normal and anomalous Green's functions are obtained diagrammatically and analytically in the condensed state of a simplest version of 3D BFM. A pairing of bosons analogous to the Cooper pairing of fermions is found. There are three coupled condensates in the model, described by the off-diagonal single-particle boson, pair-fermion and pair-boson fields. These results negate the common wisdom that the boson-fermion model is adequately described by the BCS theory at weak coupling.

PACS. 74.20.-z Theories and models of superconducting state – 74.20.Mn Nonconventional mechanisms (spin fluctuations, polarons and bipolarons, resonating valence bond model, anyon mechanism, marginal Fermi liquid, Luttinger liquid, etc.) – 74.20.Rp Pairing symmetries (other than *s*-wave) – 74.25.Dw Superconductivity phase diagrams

1 Introduction

The experimental [1–6] and theoretical [7–10] evidence for an exceptionally strong electron-phonon interaction in novel superconductors is now so overwhelming that even some advocates of the non-phononic mechanisms [11] admit the fact. A few authors (see, for example [12–15]) explored a view that the extension of the BCS theory towards the strong interaction between electrons and ion vibrations describes the phenomenon. In this regime, pairing takes place in real space due to a polaron collapse of the Fermi energy [16], or due to a low density of carriers. At first sight, bipolarons have a mass too large to be mobile. Indeed, Anderson [17] introduced small bipolarons as entirely localised objects explaining some unusual properties of chalcogenide glasses. However, it has been shown more recently that the inclusion of the on-site Coulomb repulsion leads to the favoured binding of intersite carriers [18,19]. The intersite bipolarons can tunnel with an effective mass of about 10 electron masses [15,18–20], and account for a high critical temperature [21].

Soon after Anderson [17] and Street and Mott [22] introduced localized pairs in amorphous semiconductors, a two component model of negative U centers coupled with

the Fermi sea of itinerant fermions was employed to study superconductivity in disordered metal-semiconductor alloys [23,24]. When the attractive potential U is large, the model is reduced to localized hard-core bosons spontaneously decaying into itinerant electrons and vice versa, different from a non-converting mixture of mobile charged bosons and fermions [25,26]. This boson-fermion model (BFM) was applied more generally to describe pairing electron processes with localization-delocalization [27], and a linear resistivity in the normal state of cuprates [28]. The model attracted more attention in connection with high-temperature superconductors [10,29–38], as an alternative to the bipolaron model of cuprates with intrinsically mobile bipolarons [39]. In particular, references [36,37] claimed that 2D BFM with immobile hard-core bosons is capable to reproduce some physical properties and the phase diagram of cuprates. BFM has been also adopted for a description of superfluidity of atomic fermions scattered into bound (molecular) states [40].

Most studies of BFM below its transition into a low-temperature condensed phase applied a mean-field approximation (MFA), replacing zero-momentum boson operators by *c*-numbers and neglecting the boson self-energy in the density sum rule [29,30,32,33,36–38,40]. When the bare boson energy is well above the chemical potential,

^a e-mail: a.s.alexandrov@lboro.ac.uk

the BCS ground state was found with bosons being only virtually excited [29,30]. MFA led to a conclusion that BFM exhibits features compatible with BCS characteristics [32], and describes a crossover from the BCS-like to local pair behaviour [38]. The transition was found more mean-field-like than the usual Bose condensation, i.e. characterized by a relatively small value of the fluctuation parameter Gi [33].

At the same time our previous study of BFM [34] beyond MFA revealed a crucial effect of the boson self-energy on the normal state boson spectral function and the transition temperature T_c . Reference [34] proved that the Cooper pairing of fermions via virtual bosonic states is impossible in any-dimensional BFM. It occurs only simultaneously with the Bose-Einstein condensation of real bosons. The origin of this simultaneous condensation lies in a softening of the boson mode at $T = T_c$ caused by its hybridization with fermions. The energy of zero-momentum bosons is renormalized down to *zero* at $T = T_c$, no matter how weak the boson-fermion coupling and how large the bare boson energy are [34]. One can also expect that the boson self-energy should qualitatively modify the phase transition and the whole condensed phase of BFM below T_c .

In this paper the phase transition and the condensed state of BFM are examined beyond the ordinary mean-field approximation in two and three dimensions. It is shown that $T_c = 0$ K in the two-dimensional model, even in the absence of any Coulomb repulsion, and the phase transition is never a BCS-like second-order phase transition even in 3D BFM because of the complete boson softening. A closed set of equations for fermion and boson Green's functions (GFs) is derived taking into account the self-energy effects in the condensed state of 3D BFM. There exist a boson *pair* condensate along with the fermion Cooper pair and the single-particle boson condensate in the model. Remarkably, the Gor'kov expansion [41] of GFs in the strength of the order parameter yields a zero linear term at *any* temperature below T_c , and zero upper critical field.

2 No Cooper pairing and condensation in 2D BFM

2D BFM is defined by the Hamiltonian,

$$H = \sum_{\mathbf{k}, \sigma=\uparrow, \downarrow} \xi_{\mathbf{k}} c_{\mathbf{k}, \sigma}^\dagger c_{\mathbf{k}, \sigma} + E_0 \sum_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}} + gN^{-1/2} \sum_{\mathbf{q}, \mathbf{k}} (\phi_{\mathbf{k}} b_{\mathbf{q}}^\dagger c_{-\mathbf{k}+\mathbf{q}/2, \uparrow} c_{\mathbf{k}+\mathbf{q}/2, \downarrow} + \text{H.c.}), \quad (1)$$

where $\xi_{\mathbf{k}} = -2t(\cos k_x + \cos k_y) - \mu$ is the 2D energy spectrum of fermions, $E_0 \equiv \Delta_B - 2\mu$ is the bare boson energy with respect to their chemical potential 2μ , g is the magnitude of the anisotropic hybridization interaction, $\phi_{\mathbf{k}} = \phi_{-\mathbf{k}}$ is the anisotropy factor, and N is the number of cells. Here and further I take $\hbar = c = k_B = 1$ and the lattice constant $a = 1$. Reference [36] argued

that 'superconductivity is induced in this model from the anisotropic charge-exchange interaction ($g\phi_{\mathbf{k}}$) between the conduction-band fermions and the immobile hard-core bosons', and 'the on-site Coulomb repulsion competes with this pairing' reducing the critical temperature T_c less than by 25%. Also it has been argued [37], that the calculated upper critical field of the model fits well the experimental results in cuprates.

Here I show that $T_c = 0$ K in the two-dimensional model, equation (1), even in the absence of any Coulomb repulsion, and the mean-field approximation is meaningless for any-dimensional BFM because of the complete boson softening.

Replacing boson operators by c -numbers for $\mathbf{q} = 0$ in equation (1) one obtains a linearised BCS-like equation for the fermion order-parameter (the gap function) $\Delta_{\mathbf{k}}$,

$$\Delta_{\mathbf{k}} = \frac{\tilde{g}^2 \phi_{\mathbf{k}}}{E_0 N} \sum_{\mathbf{k}'} \phi_{\mathbf{k}'} \frac{\Delta_{\mathbf{k}'} \tanh(\xi_{\mathbf{k}'}/2T)}{2\xi_{\mathbf{k}'}}, \quad (2)$$

with the coupling constant $\tilde{g}^2 = g^2(1 - 2n^B)$, renormalized by the hard-core effects. Using a two-particle fermion vertex part in the Cooper channel one can prove that this equation is perfectly correct even beyond the conventional non-crossing approximation [34]. The problem with MFA does not stem from this BCS-like equation, but from an incorrect definition of the bare boson energy with respect to the chemical potential, $E_0(T)$. This energy is determined by the atomic density of bosons (n^B) as (Eq. (9) in Ref. [36])

$$\tanh \frac{E_0}{2T} = 1 - 2n^B. \quad (3)$$

While equation (2) is correct, equation (3) is incorrect because the boson self-energy $\Sigma_b(\mathbf{q}, \Omega)$ due to the same hybridization interaction is missing. At first sight [36] the self-energy is small in comparison to the kinetic energy of fermions, if g is small. However $\Sigma_b(0, 0)$ diverges logarithmically at zero temperature [34], no matter how weak the interaction is. Therefore it should be kept in the density sum-rule, equation (3). Introducing the boson Green's function

$$D(\mathbf{q}, \Omega) = \frac{1 - 2n^B}{i\Omega - E_0 - \Sigma_b(\mathbf{q}, \Omega)} \quad (4)$$

one must replace incorrect equation (3) by

$$-\frac{T}{N} \sum_{\mathbf{q}, n} e^{i\Omega\tau} D(\mathbf{q}, \Omega) = n^B, \quad (5)$$

where $\tau = +0$, and $\Omega = 2\pi Tn$ ($n = 0, \pm 1, \pm 2, \dots$).

The divergent (cooperon) contribution to $\Sigma_b(\mathbf{q}, \Omega)$ is given by Figure 1 [34],

$$\Sigma_b(\mathbf{q}, \Omega) = -\frac{\tilde{g}^2}{2N} \sum_{\mathbf{k}} \phi_{\mathbf{k}}^2 \times \frac{\tanh[\xi_{\mathbf{k}-\mathbf{q}/2}/(2T)] + \tanh[\xi_{\mathbf{k}+\mathbf{q}/2}/(2T)]}{\xi_{\mathbf{k}-\mathbf{q}/2} + \xi_{\mathbf{k}+\mathbf{q}/2} - i\Omega}, \quad (6)$$

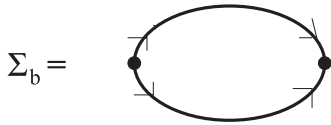


Fig. 1. Diagram for the boson self-energy. Solid lines are the fermion GFs. Vertex (dot) corresponds to the hybridization interaction.

so that one obtains

$$\Sigma_b(\mathbf{q}, 0) = \Sigma_b(0, 0) + \frac{q^2}{2M^*} + \mathcal{O}(q^4) \quad (7)$$

for small \mathbf{q} and any anisotropy factor compatible with the point-group symmetry of the cuprates. Here M^* is the boson mass, calculated analytically in reference [34] for the isotropic exchange interaction and parabolic fermion band dispersion (see also Ref. [35]). The BCS-like equation (2) has a nontrivial solution for $\Delta_{\mathbf{k}}$ at $T = T_c$, if

$$E_0 = -\Sigma_b(0, 0). \quad (8)$$

Substituting equations (7, 8) into the sum-rule, equation (5), one obtains a logarithmically divergent integral with respect to \mathbf{q} , and

$$T_c = \frac{\text{const.}}{\int_0 d q/q} = 0. \quad (9)$$

The devastating result, equation (9) is a direct consequence of the well-known theorem, which states that BEC is impossible in 2D.

One may erroneously believe that MFA results [36, 37] can be still applied in three-dimensions, where BEC is possible. However, increasing dimensionality does not make MFA a meaningful approximation for the boson-fermion model. This approximation leads to a naive conclusion that a BCS-like superconducting state occurs below the critical temperature $T_c \simeq \mu \exp(-E_0/z_c)$ via fermion pairs being *virtually* excited into *unoccupied* bosonic states [29, 30]. Here $z_c = \tilde{g}^2 N(0)$ and $N(0)$ is the density of states (DOS) in the fermionic band near the Fermi level μ . However, the Cooper pairing of fermions is impossible via virtual unoccupied bosonic states also in 3D BFM. Indeed, equations (2, 8) do not depend on the dimensionality, so that the analytical continuation of equation (4) to real frequencies ω yields the partial boson DOS as $\rho(\omega) = (1 - 2n_B)\delta(\omega)$ at $T = T_c$ and $\mathbf{q} = 0$ in any-dimensional BFM. The Cooper pairing may occur only simultaneously with the Bose-Einstein condensation of real bosons in the exact theory of 3D BFM [34]. The origin of the simultaneous condensation of the fermionic and bosonic fields in 3D BFM lies in the softening of the boson mode at $T = T_c$ caused by its hybridization with fermions.

Taking into account the boson damping and dispersion shows that the boson spectrum significantly changes for all momenta. Continuing the self-energy, equation (6) to real frequencies yields the damping (i.e. the imaginary part of the self-energy) as [34]

$$\gamma(\mathbf{q}, \omega) = \frac{\pi z_c}{4q\xi} \ln \left[\frac{\cosh(q\xi + \omega/(4T_c))}{\cosh(-q\xi + \omega/(4T_c))} \right], \quad (10)$$

where $\xi = v_F/(4T_c)$ is a coherence length, and v_F is the Fermi velocity. The damping is significant when $q\xi \ll 1$. In this region $\gamma(\mathbf{q}, \omega) = \omega\pi z_c/(8T_c)$ is comparable or even larger than the boson energy ω . Hence bosons look like overdamped diffusive modes, rather than quasiparticles in the long-wave limit [34, 35], contrary to the erroneous conclusion of reference [31], that there is ‘the onset of coherent free-particle-like motion of the bosons’ in this limit. Only outside the long-wave region, the damping becomes small. Indeed, using equation (10) one obtains $\gamma(\mathbf{q}, \omega) = \omega\pi z_c/(2qv_F) \ll \omega$, so that bosons at $q \gg 1/\xi$ are well defined quasiparticles with a logarithmic dispersion, $\omega(q) = z_c \ln(q\xi)$ [34]. Hence the boson energy disperses over the whole energy interval from zero up to E_0 .

The main mathematical problem with MFA in 3D also stems from the density sum rule, equation (5) which determines the chemical potential of the system and consequently the bare boson energy $E_0(T)$ as a function of temperature. In the framework of MFA one takes the bare boson energy in equation (2) as a temperature independent parameter, $E_0 = \tilde{g}^2 N(0) \ln(\mu/T_c)$ [33], or determines it from the conservation of the total number of particles neglecting the boson self-energy, equation (3) [30, 36, 38, 40]. Then equation (2) looks like the conventional linearized Ginzburg-Landau-Gor’kov equation [41] with a negative coefficient $\alpha \propto T - T_c$ at $T < T_c$ in the linear term. Then one concludes that the phase transition is almost the conventional BCS-like transition, at least at $E_0 \gg T_c$ [29, 30, 33]. These findings are mathematically and physically erroneous. Indeed, the term of the sum in equation (5) with $\Omega_n = 0$ is given by the integral

$$T \int \frac{d\mathbf{q}}{2\pi^3} \frac{1}{E_0 + \Sigma_b(\mathbf{q}, 0)}. \quad (11)$$

The integral converges, if and only if $E_0 \geq -\Sigma_b(0, 0)$. In fact,

$$E_0 + \Sigma_b(0, 0) = 0 \quad (12)$$

is strictly zero in the Bose-condensed state, because $\mu_b = -[E_0 + \Sigma_b(0, 0)]$ corresponds to the boson chemical potential relative to the lower edge of the boson energy spectrum. More generally, $\mu_b = 0$ corresponds to the appearance of the Bogoliubov-Goldstone mode due to a broken symmetry below T_c . This exact result makes the BSC equation (2) simply an identity [34] with $\alpha(T) \equiv 0$ at any temperature below T_c . On the other hand, MFA violates the density sum-rule, predicting the wrong negative $\alpha(T)$ below T_c .

Since $\alpha(T) = 0$, one may expect that the conventional upper critical field, $H_{c2}(T)$ is zero in BFM. To determine $H_{c2}(T)$ and explore the condensed phase of 3D BFM, one can apply the Gor’kov formalism [41], as described below.

3 Normal and anomalous Green’s functions of 3D BFM: pairing of bosons

Let us now explore a simplified version of 3D BFM in an external magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ neglecting the

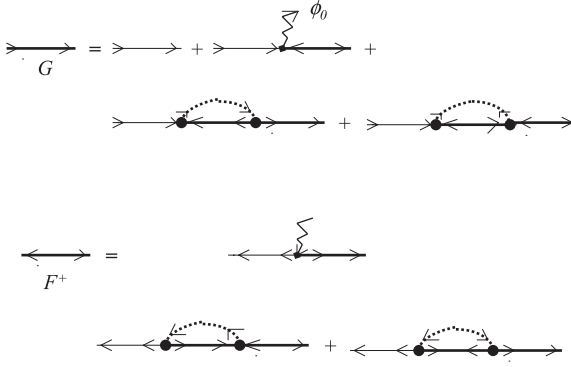


Fig. 2. Diagrams for the normal and anomalous fermion GFs. Zig-zag arrows represent the single-particle Bose condensate ϕ_0 , dotted lines are the boson GFs.

hard-core effects,

$$H = \int d\mathbf{r} \sum_s \psi_s^\dagger(\mathbf{r}) \hat{h}(\mathbf{r}) \psi_s(\mathbf{r}) + g[\phi(\mathbf{r}) \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) + \text{H.c.}] + E_0 \phi^\dagger(\mathbf{r}) \phi(\mathbf{r}), \quad (13)$$

where $\psi_s(\mathbf{r})$ and $\phi(\mathbf{r})$ are fermionic and bosonic fields, $s = \uparrow, \downarrow$ is the spin, $\hat{h}(\mathbf{r}) = -[\nabla + ie\mathbf{A}(\mathbf{r})]^2/(2m) - \mu$ is the fermion kinetic energy operator. Here the volume of the system is taken as $V = 1$.

The Matsubara field operators, $Q = \exp(H\tau)Q(\mathbf{r})\exp(-H\tau)$, $\bar{Q} = \exp(H\tau)Q^\dagger(\mathbf{r})\exp(-H\tau)$ ($Q \equiv \psi_s, \phi$) evolve with the imaginary time $-1/T \leq \tau \leq 1/T$ as

$$-\frac{\partial \psi_\uparrow(\mathbf{r}, \tau)}{\partial \tau} = \hat{h}(\mathbf{r}) \psi_\uparrow(\mathbf{r}, \tau) + g\phi(\mathbf{r}, \tau) \bar{\psi}_\downarrow(\mathbf{r}, \tau), \quad (14)$$

$$\frac{\partial \bar{\psi}_\downarrow(\mathbf{r}, \tau)}{\partial \tau} = \hat{h}^*(\mathbf{r}) \bar{\psi}_\downarrow(\mathbf{r}, \tau) - g\bar{\phi}(\mathbf{r}, \tau) \psi_\uparrow(\mathbf{r}, \tau), \quad (15)$$

$$-\frac{\partial \phi(\mathbf{r}, \tau)}{\partial \tau} = E_0 \phi(\mathbf{r}, \tau) + g\psi_\downarrow(\mathbf{r}, \tau) \psi_\uparrow(\mathbf{r}, \tau). \quad (16)$$

The theory of the condensed state can be formulated with the normal and anomalous fermion GFs [41], $\mathcal{G}(\mathbf{r}, \mathbf{r}', \tau) = -\langle T_\tau \psi_s(\mathbf{r}, \tau) \bar{\psi}_s(\mathbf{r}', 0) \rangle$, $\mathcal{F}^+(\mathbf{r}, \mathbf{r}', \tau) = \langle T_\tau \bar{\psi}_\downarrow(\mathbf{r}, \tau) \bar{\psi}_\uparrow(\mathbf{r}', 0) \rangle$, respectively, where the operation T_τ performs the time ordering. Fermionic and bosonic fields condense simultaneously [34]. Following Bogoliubov [42] the bosonic condensate is described by separating a large matrix element $\phi_0(\mathbf{r})$ in $\phi(\mathbf{r}, \tau)$ as a number, while the remaining part $\tilde{\phi}(\mathbf{r}, \tau)$ describes a supracondensate field, $\phi(\mathbf{r}, \tau) = \phi_0(\mathbf{r}) + \tilde{\phi}(\mathbf{r}, \tau)$. Then using equation (16) one obtains

$$g\phi_0(\mathbf{r}) = \Delta(\mathbf{r}) \equiv -\frac{g^2}{E_0} \mathcal{F}(\mathbf{r}, \mathbf{r}, 0+), \quad (17)$$

where $\mathcal{F}(\mathbf{r}, \mathbf{r}', \tau) = \langle T_\tau \psi_\downarrow(\mathbf{r}, \tau) \psi_\uparrow(\mathbf{r}', 0) \rangle$. The equations for GFs are obtained by using equations (14–16) and the diagrammatic technique [43] in the framework of the non-crossing approximation [44], as shown in Figures 2 and 3.

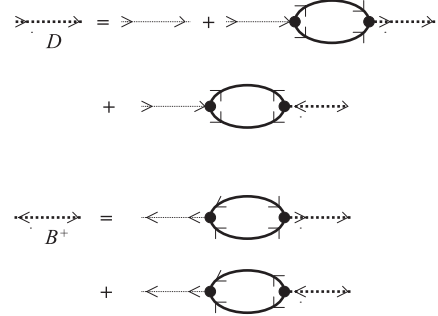


Fig. 3. Diagrams for the supracondensate boson GFs. The Cooper-pairing of fermions leads to the Cooper-pair-like boson condensate, described by the boson anomalous GF, \mathcal{B}^+ .

An important novel feature of BFM is a pairing of supracondensate bosons, caused by their hybridization with the fermionic condensate, as follows from the last diagram in Figure 3. Hence, one has to introduce an *anomalous* supracondensate boson GF, $\mathcal{B}^+(\mathbf{r}, \mathbf{r}', \tau) = \langle T_\tau \tilde{\phi}(\mathbf{r}, \tau) \tilde{\phi}(\mathbf{r}', 0) \rangle$ along with the normal boson GF, $\mathcal{D}(\mathbf{r}, \mathbf{r}', \tau) = -\langle T_\tau \tilde{\phi}(\mathbf{r}, \tau) \tilde{\phi}(\mathbf{r}', 0) \rangle$. The diagrams, Figures 2 and 3, are transformed into analytical equations for the time Fourier-components of the fermion GFs with the Matsubara frequencies $\omega = \pi T(2n + 1)$ ($n = 0, \pm 1, \pm 2, \dots$) as

$$[i\omega - \hat{h}(\mathbf{r})] \mathcal{G}_\omega(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') - \Delta(\mathbf{r}) \mathcal{F}_\omega^+(\mathbf{r}, \mathbf{r}') - g^2 T \sum_{\omega'} \int d\mathbf{x} \mathcal{G}_{-\omega'}(\mathbf{x}, \mathbf{r}) \mathcal{D}_{\omega-\omega'}(\mathbf{r}, \mathbf{x}) \mathcal{G}_\omega(\mathbf{x}, \mathbf{r}') - g^2 T \sum_{\omega'} \int d\mathbf{x} \mathcal{F}_{\omega'}^+(\mathbf{r}, \mathbf{x}) \mathcal{B}_{\omega+\omega'}(\mathbf{r}, \mathbf{x}) \mathcal{F}_\omega^+(\mathbf{x}, \mathbf{r}'), \quad (18)$$

$$[-i\omega - \hat{h}^*(\mathbf{r})] \mathcal{F}_\omega^+(\mathbf{r}, \mathbf{r}') = \Delta^*(\mathbf{r}) \mathcal{G}_\omega(\mathbf{r}, \mathbf{r}') - g^2 T \sum_{\omega'} \int d\mathbf{x} \mathcal{G}_{\omega'}(\mathbf{r}, \mathbf{x}) \mathcal{D}_{\omega'-\omega}(\mathbf{x}, \mathbf{r}) \mathcal{F}_\omega^+(\mathbf{x}, \mathbf{r}') + g^2 T \sum_{\omega'} \int d\mathbf{x} \mathcal{F}_{-\omega'}(\mathbf{r}, \mathbf{x}) \mathcal{B}_{-\omega-\omega'}^+(\mathbf{r}, \mathbf{x}) \mathcal{G}_\omega(\mathbf{x}, \mathbf{r}')$$

and,

$$(i\Omega - E_0) \mathcal{D}_\Omega(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') - g^2 T \sum_{\omega'} \int d\mathbf{x} \mathcal{G}_{\omega'}(\mathbf{r}, \mathbf{x}) \mathcal{G}_{\Omega-\omega'}(\mathbf{r}, \mathbf{x}) \mathcal{D}_\Omega(\mathbf{x}, \mathbf{r}') - g^2 T \sum_{\omega'} \int d\mathbf{x} \mathcal{F}_{\omega'}(\mathbf{r}, \mathbf{x}) \mathcal{F}_{\Omega-\omega'}(\mathbf{r}, \mathbf{x}) \mathcal{B}_\Omega^+(\mathbf{x}, \mathbf{r}'), \quad (19)$$

$$(-i\Omega - E_0) \mathcal{B}_\Omega^+(\mathbf{r}, \mathbf{r}') = g^2 T \sum_{\omega'} \int d\mathbf{x} \mathcal{F}_{-\omega'}^+(\mathbf{r}, \mathbf{x}) \mathcal{F}_{-\Omega+\omega'}^+(\mathbf{r}, \mathbf{x}) \mathcal{D}_\Omega(\mathbf{x}, \mathbf{r}') - g^2 T \sum_{\omega'} \int d\mathbf{x} \mathcal{G}_{-\omega'}(\mathbf{x}, \mathbf{r}) \mathcal{G}_{\omega'-\Omega}(\mathbf{x}, \mathbf{r}) \mathcal{B}_\Omega^+(\mathbf{x}, \mathbf{r}')$$

for the boson GFs with $\mathcal{B}(\mathbf{r}, \mathbf{r}', \tau) = \langle T_\tau \tilde{\phi}(\mathbf{r}, \tau) \tilde{\phi}(\mathbf{r}', 0) \rangle$.

4 Gor'kov expansion

These equations can be formally solved in the homogeneous case without the external field, $\mathbf{A} = 0$. Transforming into the momentum space yields GFs' time-space Fourier components as

$$\mathcal{G}(\mathbf{k}, \omega) = -\frac{i\tilde{\omega}^* + \xi_{\mathbf{k}}}{|i\tilde{\omega} - \xi_{\mathbf{k}}|^2 + |\tilde{\Delta}(\mathbf{k}, \omega)|^2}, \quad (20)$$

$$\mathcal{F}^+(\mathbf{k}, \omega) = \frac{\tilde{\Delta}^*(\mathbf{k}, \omega)}{|i\tilde{\omega} - \xi_{\mathbf{k}}|^2 + |\tilde{\Delta}(\mathbf{k}, \omega)|^2}, \quad (21)$$

and

$$\mathcal{D}(\mathbf{q}, \Omega) = -\frac{i\tilde{\Omega}^* + E_0}{|i\tilde{\Omega} - E_0|^2 + |\Gamma(\mathbf{q}, \Omega)|^2}, \quad (22)$$

$$\mathcal{B}^+(\mathbf{q}, \Omega) = \frac{\Gamma^*(\mathbf{q}, \Omega)}{|i\tilde{\Omega} - E_0|^2 + |\Gamma(\mathbf{q}, \Omega)|^2}, \quad (23)$$

where $\tilde{\omega} \equiv \omega + i\Sigma_f(\mathbf{k}, \omega)$, $\tilde{\Omega} \equiv \Omega + i\Sigma_b(\mathbf{q}, \Omega)$, and $\xi_{\mathbf{k}} = k^2/(2m) - \mu$. The fermionic order parameter, renormalised with respect to the mean-field Δ due to the formation of the boson-pair condensate, is given by

$$\tilde{\Delta}(\mathbf{k}, \omega) = \Delta + g^2 T \sum_{\omega'} \int \frac{d\mathbf{q}}{2\pi^3} \mathcal{F}^+(\mathbf{k} - \mathbf{q}, \omega') \mathcal{B}(\mathbf{q}, \omega + \omega'), \quad (24)$$

and the boson-pair order parameter, generated by the hybridization with the fermion Cooper pairs, is

$$\Gamma(\mathbf{q}, \Omega) = g^2 T \sum_{\omega'} \int \frac{d\mathbf{k}}{2\pi^3} \mathcal{F}(\mathbf{k}, \omega') \mathcal{F}(\mathbf{q} - \mathbf{k}, \Omega - \omega'). \quad (25)$$

Hence, there are three coupled condensates in the model described by the off-diagonal fields $g\phi_0$, $\tilde{\Delta}$, and Γ , rather than two, as in MFA. At low temperatures all of them have about the same magnitude, as the fermion, Figure 4, and boson, Figure 1, self-energies,

$$\Sigma_f(\mathbf{k}, \omega) = -g^2 T \sum_{\omega'} \int \frac{d\mathbf{q}}{2\pi^3} \mathcal{G}(\mathbf{q} - \mathbf{k}, -\omega') \mathcal{D}(\mathbf{q}, \omega - \omega'), \quad (26)$$

$$\Sigma_b(\mathbf{q}, \Omega) = -g^2 T \sum_{\omega'} \int \frac{d\mathbf{k}}{2\pi^3} \mathcal{G}(\mathbf{k}, \omega') \mathcal{G}(\mathbf{q} - \mathbf{k}, \Omega - \omega'), \quad (27)$$

respectively.

On the other hand, when the temperature is close to T_c (i.e. $T_c - T \ll T_c$), the boson pair condensate is weak compared with the single-particle boson and the Cooper pair condensates. In this temperature range Γ , equation (25) is of the second order in Δ , $\Gamma \propto \Delta^2$, so that the anomalous boson GF can be neglected, since Δ is small. The fermion self-energy, equation (26) is a regular function of ω and \mathbf{k} , so that it can be absorbed in the renormalized fermion band dispersion. Then the fermion normal and anomalous GFs, equations (20, 21) look like the familiar GFs of the BCS theory, and one can apply the Gor'kov expansion [41] in powers of $\Delta(\mathbf{r})$ to describe the condensed

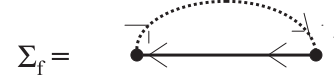


Fig. 4. The fermion self-energy.

phase of BFM in the magnetic field near the transition. Using equation (18) one obtains to the terms linear in Δ

$$\Delta^*(\mathbf{r}) = \frac{g^2}{E_0} T \sum_{\omega_n} \int d\mathbf{x} \mathcal{G}_{-\omega_n}^{(n)}(\mathbf{x}, \mathbf{r}) \Delta^*(\mathbf{x}) \mathcal{G}_{\omega_n}^{(n)}(\mathbf{x}, \mathbf{r}). \quad (28)$$

The spatial variations of the vector potential are small near the transition. If $\mathbf{A}(\mathbf{r})$ varies slowly, the normal state GF, $\mathcal{G}_{\omega}^{(n)}(\mathbf{r}, \mathbf{r}')$ differs from the zero-field normal state GF, $\mathcal{G}_{\omega}^{(0)}(\mathbf{r} - \mathbf{r}')$ only by a phase [41] $\mathcal{G}_{\omega}^{(n)}(\mathbf{r}, \mathbf{r}') = \exp[-ie\mathbf{A}(\mathbf{r}) \cdot (\mathbf{r} - \mathbf{r}')] \mathcal{G}_{\omega}^{(0)}(\mathbf{r} - \mathbf{r}')$. Expanding all quantities near the point $\mathbf{x} = \mathbf{r}$ in equation (28) up to the second order in $\mathbf{x} - \mathbf{r}$ inclusive, one obtains the linearised equation for the fermionic order parameter as

$$\gamma[\nabla - 2ie\mathbf{A}(\mathbf{r})]^2 \Delta(\mathbf{r}) = \alpha \Delta(\mathbf{r}), \quad (29)$$

where

$$\alpha = 1 + \frac{\Sigma_b(0, 0)}{E_0} \approx 1 - \frac{g^2 N(0)}{E_0} \ln \frac{\mu}{T}, \quad (30)$$

and $\gamma \approx 7\zeta(3)v_F^2 g^2 N(0)/(48\pi^2 T^2 E_0)$.

5 Conclusion

The coefficient $\alpha(T)$ disappears in equation (29), since $E_0 = -\Sigma_b(0, 0)$ at and below T_c , equation (12). It means that the phase transition is never a BCS-like second-order phase transition even at large E_0 and small g . In fact, the transition is driven by the Bose-Einstein condensation of *real* bosons with $\mathbf{q} = 0$, which occur due to the complete softening of their spectrum at T_c . Remarkably, the conventional upper critical field, determined as the field, where a non-trivial solution of the *linearised* Gor'kov equation (29) occurs, is zero in BFM, $H_{c2}(T) = 0$. It is not a finite $H_{c2}(T)$ found in reference [37] using MFA.

This qualitative failure of MFA might be rather unexpected, if one believes that bosons in equation (1) play the same role as phonons in the BCS superconductor. This is not the case for two reasons. The first one is the density sum-rule, equation (5), for bosons which is not applied to phonons. The second being that the boson self-energy is given by the divergent (at $T = 0$) Cooperon diagram, while the self-energy of phonons is finite at small coupling.

In the homogeneous case $\Delta(T)$ should be determined from equation (5) rather than from the BCS-like equation (2), which is actually the identity. To get an insight into the magnetic properties of the condensed phase one has to solve equations (18, 19) and equation (5) keeping the non-linear terms. Even at temperatures well below T_c the condensed state is fundamentally different from the

MFA ground state, because of the pairing of bosons. The latter is similar to the Cooper-like pairing of supracondensate ^4He atoms [45], proposed as an explanation of the small density of the single-particle Bose condensate in superfluid Helium-4. The pair-boson condensate should significantly modify the thermodynamic properties of the condensed BFM compared with the MFA predictions. The common wisdom that at weak coupling the boson-fermion model is adequately described by the BCS theory, is therefore negated by our theory.

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